

On linearized gravity in the Randall-Sundrum scenario

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Abstract

In the literature about the Randall-Sundrum scenario one finds on one hand that there exist (small) corrections to Newton's law of gravity on the brane, and on another that the exact (and henceforth linearized) Einstein equations can be recovered on the brane. The explanation for these seemingly contradictory results is that the behaviour of the bulk far from the brane is different in both models. We show that explicitly in this paper.

I. Introduction

There has been recently an increasing interest for gravity theories within spacetimes with large extra dimensions and the idea that our universe may be a four dimensional singular hypersurface, or “brane”, in a five dimensional spacetime, or “bulk”. The Randall-Sundrum scenario [1], where our universe is a four dimensional quasi-Minkowskian edge of a double-sided perturbed anti-de Sitter spacetime, was the first explicit model where the linearized Einstein equations were found to hold on the brane, apart from small $1/r^2$ corrections to Newton’s potential. This claim was thoroughly analyzed and the corrections to Newton’s law exactly calculated [2]. Cosmological models were then built, where the brane is taken to be a Robertson-Walker spacetime embedded in an anti-de Sitter bulk [3]. The perturbations of these models, in the view of calculating the microwave background anisotropies, are currently being studied and compared to the perturbations of standard, four dimensional, Friedmann universes [4-5].

In all these papers, which deal with linear perturbations of either a Minkowski or a Robertson-Walker brane in an anti-de Sitter bulk, corrections to the standard linearized four dimensional Einstein equations are found and analyzed.

On another hand, there exist a number of papers which study under which conditions on the brane and the bulk one can or cannot recover the exact (and henceforth linearized) four dimensional Einstein equations on the brane. For example Chamblin Hawking and Reall [6] showed that the exact Schwarzschild solution can hold on a brane. The question was studied more generally in e.g. [7-8].

Now, if the exact Schwarzschild solution can hold on a brane, a fortiori the linearized Schwarzschild solution and hence Newton’s law can hold as well, and this is in contradiction with the result that there should be $1/r^2$ corrections to Newton’s potential. A possible explanation for these seemingly contradictory results* is that the Chamblin-Hawking-Reall solution exhibits a string-like curvature singularity on a line perpendicular to the brane whereas the Randall-Sundrum solution is regular everywhere in the bulk. This is true but the question then becomes : how did Randall-Sundrum find only one solution (well behaved in the bulk and exhibiting a $1/r^2$ correction to Newton’s law in the brane) and did not find a whole family of solutions, among which theirs plus the linearized version of the Chamblin-Hawking-Reall solution which is strictly Newtonian on the brane.

The answer obviously lies in an analysis of the boundary conditions far away from the brane which the various authors choose for the perturbations. Since some authors [1-2] use a field theoretic approach to study the perturbation equations, based on Green’s functions and retarded propagators, whereas some others [5-7] use a geometric approach which stems from the theory of thin shells in general relativity, it is not completely straightforward to see where the two approaches differ and where different choices of boundary conditions are made. It is the purpose of this paper to reconcile the two points of view.

In a first step (Section II) we analyze the example of a massless scalar field in a five dimensional Minkowski bulk which is simpler than, but akin to, the Randall-Sundrum scenario. We show that the standard field theoretic approach yields a solution which is well-behaved far away in the bulk but which is non-“newtonian” on the brane. We then

* Jaume Garriga and Keichi Maeda, private communications

show that the price to pay to get a “newtonian” solution on the brane is to choose a solution which diverges at infinity in the bulk.

In Section III we show that the situation is similar in the Randall-Sundrum scenario, with the important difference that perturbations which diverge at infinity in the bulk must not be necessarily rejected as their divergence may be only a gauge effect. Indeed, we show in Section IV that the well-behaved (at least outside the source) linearized Chamblin-Hawking-Reall solution corresponds to perturbations which diverge in the conformally Minkowskian coordinate system used in the field theoretic approach. Finally, in Section V, we rederive the now standard $1/r^2$ correction to Newton’s law obtained in [1-2] but with an emphasis on the choice of boundary condition which yields the result.

We draw a few conclusions in Section VI.

II. The scalar field analogy

Consider a 5-dimensional Minkowski space-time in Minkowskian coordinates $(x^0, \vec{r} = (x^1, x^2, x^3), y)$; cut it along the timelike hypersurface $y = 0$; make a copy of the $y \geq 0$ region ; paste it along the original and get a “ Z_2 -symmetric bulk”, that is a double-sided “half” 5-D Minkowski space-time.

Consider now a massless scalar field $\Phi(x^0, \vec{r}, y)$ in this bulk, obeying the Klein-Gordon equation

$$\square_5 \Phi = 0 \tag{2.1}$$

everywhere but on the “brane” $y = 0$, the boundary condition being

$$\partial_y \Phi|_0 = \alpha \delta_3(\vec{r}) \tag{2.2}$$

with α a constant.

In the analogy we are developping here, the 5-D Minkowski bulk replaces the anti-de Sitter bulk of the Randall-Sundrum scenario ; Equation (2.1) replaces the linearized Einstein equations governing the perturbations of the anti-de Sitter bulk ; the hypersurface $y = 0$ represents the brane ; and equation (2.2) replaces the Israel junction conditions, in the case when “matter” on the brane is a static, point-like source.

Since we consider a static source, we shall look for a static solution of (2.1).

A standard way to solve (2.1) and (2.2) in the static case is to replace them by the equation, valid for all \vec{r} and all y (positive, zero or negative)

$$\triangle_4 \Phi(\vec{r}, y) = 2\alpha \delta_3(\vec{r}) \delta(y) = 2\alpha \delta_4(\vec{r}, y) \tag{2.3}$$

which embodies the bulk “ Z_2 -symmetry”. (This is how Garriga and Tanaka [2] for example solve the corresponding equations of the Randall-Sundrum scenario.) Equation (2.3) is

easily solved within the theory of distributions and yields*

$$\Phi(\vec{r}, y) = -\frac{\alpha}{2\pi^2} \frac{1}{(r^2 + y^2)}. \quad (2.4)$$

On the brane, the solution $\Phi(\vec{r}, y = 0) \propto 1/r^2$ is “non-newtonian” (in the Randall-Sundrum set up the corresponding solution will incorporate the $1/r^2$ corrections to Newton’s law) ; as for $\partial_y \Phi|_0$, it is zero everywhere, but at $\vec{r} = 0$ where it is a delta function (see (2.2)). Near the brane, the expansion of (2.4), in a distributional sense, is

$$\Phi(\vec{r}, y) = -\frac{\alpha}{2\pi^2} \frac{1}{r^2} + \alpha \delta_3(\vec{r}) y + \frac{\alpha}{2\pi^2} \frac{1}{r^4} y^2 - \alpha \Delta \delta_3(\vec{r}) \frac{y^3}{6} + \mathcal{O}(y^4) \quad (2.5)$$

and exhibits delta-like singularities at $\vec{r} = 0$ for all $y \neq 0$ which are an artifact of the expansion since they disappear when the series is summed to give back (2.4).

Before trying to see how one could get a “newtonian” $1/r$ solution on the brane, let us solve the equations (2.1) and (2.2) in a slightly different way. Everywhere outside the brane, and for $y > 0$, equation (2.1) reduces, in the static case, to $\Delta_4 \Phi(\vec{r}, y) = 0$. Let us solve this equation in Fourier space : $\Phi(\vec{r}, y) = \int \frac{d^3 \vec{k}}{(2\pi)^{\frac{3}{2}}} e^{i\vec{k} \cdot \vec{r}} \hat{\Phi}_{\vec{k}}(y)$ with

$$-k^2 \hat{\Phi}_{\vec{k}} + \partial_{yy}^2 \hat{\Phi}_{\vec{k}} = 0. \quad (2.6)$$

(In the Randall-Sundrum scenario this equation will become a Bessel equation.) This equation has a well behaved solution at $y \rightarrow +\infty$ which is

$$\hat{\Phi}_{\vec{k}} = \frac{b_{\vec{k}}}{(2\pi)^{\frac{3}{2}}} e^{-ky}. \quad (2.7)$$

One now imposes the boundary condition (2.2) which gives : $\partial_y \hat{\Phi}_{\vec{k}}|_0 = \alpha/(2\pi)^{\frac{3}{2}}$, that is $b_{\vec{k}} = -\frac{\alpha}{k}$. Hence $\Phi(\vec{r}, y) = -\alpha \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{k} e^{(i\vec{k} \cdot \vec{r} - ky)} = -\frac{\alpha}{2\pi^2} \frac{1}{r^2 + y^2}$. In thus proceeding we see that the “standard” solution (2.4) is the one which corresponds to Fourier modes $\hat{\Phi}_{\vec{k}}(y)$ which converge when $y \rightarrow +\infty$.

Let us now see what scalar field we must choose in the bulk in order to recover a “newtonian” $1/r$ solution on the brane. In order to do so we impose

$$\Delta_3 \Phi|_0 = \alpha \delta_3(\vec{r}) \quad \implies \quad \Phi(\vec{r}, y = 0) = -\frac{\alpha}{4\pi} \frac{1}{r}. \quad (2.8)$$

* One can also solve this equation in Fourier space. One decomposes $\Phi(\vec{r}, y)$ as $\Phi(\vec{r}, y) = \int \frac{d^3 \vec{k}}{(2\pi)^{\frac{3}{2}}} e^{i\vec{k} \cdot \vec{r}} \hat{\Phi}_{\vec{k}}(y)$. Using the fact that the Fourier transform of $\delta_n(\vec{r})$ is $(2\pi)^{-\frac{n}{2}}$ the equation for $\hat{\Phi}_{\vec{k}}(y)$ is : $-k^2 \hat{\Phi}_{\vec{k}} + \partial_{yy}^2 \hat{\Phi}_{\vec{k}} = \frac{2\alpha}{(2\pi)^{\frac{3}{2}}} \delta(y)$. The solution is obtained by further decomposing $\hat{\Phi}_{\vec{k}}(y)$ in a “tower of massive modes” as $\hat{\Phi}_{\vec{k}}(y) = \int \frac{dm}{\sqrt{2\pi}} e^{imy} \bar{\Phi}_{\vec{k}, m}$ with $\bar{\Phi}_{\vec{k}, m} = -\frac{2\alpha}{(2\pi)^2} \frac{1}{k^2 + m^2}$. Hence the solution, after regularization : $\Phi(\vec{r}, y) = -\frac{\alpha}{2\pi^2} \frac{1}{r^2 + y^2}$.

Then the junction condition (2.2) gives the first y derivative of $\Phi(\vec{r}, y)$ on the brane ; as for the bulk equation (2.1) together with (2.8) it gives the second y derivative, so that we get by iteration and for $y \geq 0$

$$\Phi(\vec{r}, y) = \alpha \left[-\frac{1}{4\pi r} + \delta_3(\vec{r}) \left(y - \frac{y^2}{2} \right) - \Delta_3 \delta_3(\vec{r}) \left(\frac{y^3}{6} - \frac{y^4}{24} \right) + \mathcal{O}(y^5) \right] \quad (2.9)$$

which is equal to the “newtonian” solution everywhere outside the source $\vec{r} = 0$. (In the Randall-Sundrum set up this solution will become the expansion in y , for $\vec{r} \neq 0$, of the linearized Chamblin-Hawking-Reall solution.) In Fourier space we have, using the fact that the Fourier transform of $1/4\pi r$ is $[(2\pi)^{3/2}k^2]^{-1}$

$$\hat{\Phi}_{\vec{k}}(y) = \frac{\alpha}{(2\pi)^{\frac{3}{2}}} \left[-\frac{1}{k^2} + \left(y - \frac{y^2}{2} \right) + k^2 \left(\frac{y^3}{6} - \frac{y^4}{24} \right) + \mathcal{O}(y^5) \right]. \quad (2.10)$$

Now $\hat{\Phi}_{\vec{k}}(y)$ is the solution of (2.6) which coincides with the expansion (2.10) near the brane. One hence readily obtains $\hat{\Phi}_{\vec{k}}(y)$ in closed form as

$$\hat{\Phi}_{\vec{k}}(y) = \frac{\alpha}{2(2\pi)^{\frac{3}{2}}} \frac{1}{k^2} [(k-1)e^{ky} - (k+1)e^{-ky}] \quad (2.11)$$

which diverges exponentially as $ky \rightarrow +\infty$, so that its Fourier transform $\Phi(\vec{r}, y)$ (whose expansion near the brane is given by (2.9)) is not a priori defined.

We therefore see on this scalar field example that the price to pay in order to get a “newtonian” solution on the brane is that we must choose a solution whose Fourier transform diverges at infinity in the bulk. The situation will be similar in the Randall-Sundrum scenario, with the very important difference that perturbations which diverge at infinity in the bulk will not have to be necessarily rejected as their divergence may be only a gauge effect.

III. The equations for (static) gravity on a quasi-Minkowskian brane embedded in a perturbed anti-de Sitter bulk

We use conformally Minkowskian coordinates $X^A = \{x^\mu = [x^0, \vec{r} = (x^1, x^2, x^3)], w\}$ to describe a five dimensional perturbed anti-de Sitter spacetime \mathcal{V}_5 . We write the metric as

$$ds^2|_5 = \mathcal{G}_{AB} dX^A dX^B \quad \text{with} \quad \mathcal{G}_{AB} = \left(\frac{\mathcal{L}}{w} \right)^2 (\eta_{AB} + \gamma_{AB}) \quad (3.1)$$

where \mathcal{L} is a (positive) constant. \mathcal{V}_5 is taken to be an Einstein space, solution of the Einstein equations $\mathcal{R}_{AB} = -\frac{4}{\mathcal{L}^2} \mathcal{G}_{AB}$ where \mathcal{R}_{AB} is the Ricci tensor of the metric \mathcal{G}_{AB} . In the gauge

$$\gamma_{Aw} = 0 \quad , \quad \gamma_\mu^\mu = \partial_\rho \gamma_\mu^\rho = 0 \quad (3.2)$$

these equations reduce to (see e.g. [5] for details)

$$\square_4 \gamma_{\mu\nu} + \partial_{ww} \gamma_{\mu\nu} - \frac{3}{w} \partial_w \gamma_{\mu\nu} = 0. \quad (3.3)$$

We consider now in \mathcal{V}_5 the hypersurface Σ defined by

$$w = \mathcal{L} + \zeta(x^\mu) \quad (3.4)$$

where the function $\zeta(x^\mu)$ is a priori arbitrary and describes the so-called “brane-bending” effect. The induced metric on Σ is

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \quad \text{with} \quad h_{\mu\nu} = \gamma_{\mu\nu}|_\Sigma - 2\frac{\zeta}{\mathcal{L}}\eta_{\mu\nu}. \quad (3.5)$$

The Randall-Sundrum brane scenario is obtained by cutting \mathcal{V}_5 along Σ , by making a copy of the $w \geq \mathcal{L} + \zeta$ side and pasting it along Σ . The integration of Einstein’s equations across the edge, or brane, of this new manifold yields the Lanczos-Darmois-Israel equations (see e.g. [5] for details) which give the stress-energy tensor of the matter in the brane Σ as $\kappa T_\nu^\mu = -\frac{6}{\mathcal{L}}\delta_\nu^\mu + \kappa S_\nu^\mu$ with κ a coupling constant and

$$\frac{\kappa}{2} \left(S_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}S \right) = \partial_{\mu\nu}\zeta - \frac{1}{2}(\partial_w\gamma_{\mu\nu})|_\Sigma, \quad (3.6)$$

which implies

$$\square_4\zeta = -\frac{\kappa}{6}S. \quad (3.7)$$

Equations (3.3) (3.5) (3.6) and (3.7) are standard : they can be found in various guise in the literature [2].

In the following we shall concentrate on a static, point-like source

$$S_{00} = M\delta(\vec{r}) \quad , \quad S_{0i} = S_{ij} = 0. \quad (3.8)$$

Equation (3.7) then gives ζ as

$$\zeta = -\frac{\kappa M}{24\pi} \frac{1}{r}. \quad (3.9)$$

The junction condition (3.6) then gives the first w -derivative of the bulk metric on the brane as

$$\partial_w\gamma_{00}|_\Sigma = -\frac{2\kappa M}{3}\delta(\vec{r}) \quad , \quad \partial_w\gamma_{0i}|_\Sigma = 0 \quad , \quad \partial_w\gamma_{ij}|_\Sigma = -\frac{\kappa M}{3}\delta(\vec{r})\delta_{ij} - \frac{\kappa M}{12\pi}\partial_{ij}\frac{1}{r}. \quad (3.10)$$

Finally the general solution of (3.3) in the static case is a superposition of Fourier modes :

$$\gamma_{\mu\nu}(x^\mu, w) = \int \frac{d^3\vec{k}}{(2\pi)^{\frac{3}{2}}} e^{i\vec{k} \cdot \vec{r}} \hat{\gamma}_{\mu\nu}(\vec{k}, w) \quad \text{with}$$

$$\hat{\gamma}_{\mu\nu}(\vec{k}, w) = w^2 \left[e_{\mu\nu}^{(1)}(\vec{k}) H_2^{(1)}(ikw) + e_{\mu\nu}^{(2)}(\vec{k}) H_2^{(2)}(ikw) \right] \quad (3.11)$$

where (because of (3.2)) the polarization tensors $e_{\mu\nu}^{(1,2)}(\vec{k})$ are transverse and traceless (and thus have a priori five freely specifiable components), and where $H_2^{(1,2)}(ikw)$ are the Hankel functions of first and second kind and of order 2. When $w \rightarrow +\infty$, $H_2^{(2)}(ikw)$ diverges

exponentially but we do not eliminate it a priori, in keeping with the conclusions of the previous Section. The junction condition (3.10) therefore determines only a combination of the polarization tensors, to wit

$$\begin{aligned}
e_{00}^{(1)}(\vec{k})H_1^{(1)}(ik\mathcal{L}) + e_{00}^{(2)}(\vec{k})H_1^{(2)}(ik\mathcal{L}) &= -\frac{2\kappa M}{3} \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{ik\mathcal{L}^2} \\
e_{0i}^{(1)}(\vec{k})H_1^{(1)}(ik\mathcal{L}) + e_{0i}^{(2)}(\vec{k})H_1^{(2)}(ik\mathcal{L}) &= 0 \\
e_{ij}^{(1)}(\vec{k})H_1^{(1)}(ik\mathcal{L}) + e_{ij}^{(2)}(\vec{k})H_1^{(2)}(ik\mathcal{L}) &= -\frac{\kappa M}{3} \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{ik\mathcal{L}^2} \left(\delta_{ij} - \frac{k_i k_j}{4\pi k^2} \right).
\end{aligned} \tag{3.12}$$

To summarize : when matter on the brane is a static, point-like source, the metric on the brane is (3.5) with ζ given by (3.9) and $\gamma_{\mu\nu}|_\Sigma$ given by (3.11) (with $w = \mathcal{L}$) and the polarization tensors restricted by condition (3.12). In order to determine the brane metric completely we must add an extra condition, for example that Einstein's linearized equations be recovered on the brane (Section IV) or that (3.11) converge (Section V).

IV. Choosing a bulk such that Schwarzschild's linearized solution holds on the brane

In order to compare the equations (3.3) (3.5) and (3.6) which govern linearized gravity on the brane to the standard 4-D linearized Einstein equations, we find it convenient to go to harmonic coordinates in the brane. Technically this means : $x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \epsilon^\mu \Rightarrow h_{\mu\nu} \rightarrow \tilde{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$, that is using (3.5)

$$\tilde{h}_{\mu\nu} = \gamma_{\mu\nu}|_\Sigma - 2\frac{\zeta}{\mathcal{L}}\eta_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu. \tag{4.1}$$

The harmonicity condition, to wit $\partial_\mu \left(\tilde{h}_\nu^\mu - \frac{1}{2}\delta_\nu^\mu \tilde{h} \right) = 0$ imposes

$$\square_4 \epsilon_\mu = -\frac{2}{\mathcal{L}} \partial_\mu \zeta. \tag{4.2}$$

We then have, using (3.6) and (3.7)

$$\square_4 \tilde{h}_{\mu\nu} = -\frac{2}{\mathcal{L}} \kappa \left(S_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} S \right) + \square_4 \gamma_{\mu\nu}|_\Sigma - \frac{2}{\mathcal{L}} (\partial_w \gamma_{\mu\nu})|_\Sigma. \tag{4.3}$$

If the sum of the last two terms is zero the linearized Einstein equations are recovered on the brane, it being understood that $\mathcal{L}^{-1}\kappa \equiv 8\pi G$, G being Newton's constant.

Let us again concentrate on the case of a static and point-like source (3.8).

The linearized Einstein equations $\square_4 \tilde{h}_{\mu\nu} = -\frac{2}{\mathcal{L}} \kappa (S_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} S)$ then give

$$\tilde{h}_{00} = \frac{\kappa M}{4\pi\mathcal{L}} \frac{1}{r} = \frac{2GM}{r}, \quad \tilde{h}_{0i} = 0, \quad \tilde{h}_{ij} = \frac{\kappa M}{4\pi\mathcal{L}} \frac{1}{r} \delta_{ij} = \frac{2GM}{r} \delta_{ij} \tag{4.4}$$

which is nothing but the expansion at linear order of the Schwarzschild metric in harmonic coordinates.

Let us now see which bulk is required to yield (4.4). As an easy manipulation of the Bessel functions shows the junction condition (3.12) together with the condition that Einstein's linearized equations be recovered on the brane, that is

$$\Delta_3 \gamma_{\mu\nu}|_\Sigma - \frac{2}{\mathcal{L}}(\partial_w \gamma_{\mu\nu})|_\Sigma = 0$$

completely determine the polarisation vectors entering $\hat{\gamma}_{\mu\nu}(\vec{k}, w)$ in (3.11) and we have

$$\hat{\gamma}_{\mu\nu}(\vec{k}, w) = i \frac{\kappa\pi M}{12\mathcal{L}} \frac{1}{(2\pi)^{\frac{3}{2}}} w^2 \left[H_0^{(1)}(ik\mathcal{L}) H_2^{(2)}(ikw) - H_0^{(2)}(ik\mathcal{L}) H_2^{(1)}(ikw) \right] c_{\mu\nu} \quad (4.5)$$

with $c_{00} = 2$, $c_{0i} = 0$ and $c_{ij} = \delta_{ij} - k_i k_j / 4\pi k^2$.

These Fourier modes diverge exponentially when $kw \rightarrow +\infty$ so that their Fourier integral $\gamma_{\mu\nu}(\vec{r}, w)$ is not defined. However we do know the geometry of the bulk when the metric on the brane is given by (4.4). It is the expansion at linear order of the Chamblin-Hawking-Reall metric [6] which reads, in Gaussian normal coordinates (\tilde{x}^μ, y) , everywhere outside the source, that is for $\tilde{x}^i \neq 0$

$$ds^2|_5 = dy^2 + e^{-2y/\mathcal{L}} \left[- \left(1 - \frac{2GM}{r} \right) (d\tilde{x}^0)^2 + \left(1 + \frac{2GM}{r} \right) \delta_{ij} d\tilde{x}^i d\tilde{x}^j \right]. \quad (4.6)$$

Therefore the badly divergent metric (4.5) must turn into the good-looking metric (4.6), at least outside the source, when one goes from the conformally Minkowskian coordinates (\tilde{x}^μ, w) to the Gaussian coordinates (\tilde{x}^μ, y) . This is indeed what happens, as is shown in the Appendix.

The conclusion which we can draw from this Section is that divergent solutions of the anti-de Sitter perturbation equations (3.3) must perhaps not be a priori be rejected as they can in some cases be transformed into “regular” ones by a mere change of coordinates. We write “regular” with inverted commas because the Chamblin-Hawking-Reall metric possesses a curvature singularity at $\vec{r} = 0$, for all y , and may have to be rejected on these grounds. But this is a separate issue from the one we are discussing here which deals with the properties of the bulk far away from the brane.

V. On the $1/r^2$ correction to Newton's law

We just saw that in order to recover Einstein's linearized equations (and hence Newton's) on the brane one must allow for divergent perturbations in the bulk, at least when using conformally Minkowskian coordinates. We show in this Section that the now standard $1/r^2$ correction to Newton's law is due to imposing perturbations in the bulk which remain small in conformally Minkowskian coordinates.

The “standard” way to study linearized gravity on a quasi-Minkowskian brane is to combine the perturbation equation (3.3) and the junction condition (3.5) into the single equation, valid for all w

$$\square_4 \gamma_{\mu\nu} + \partial_{ww} \gamma_{\mu\nu} - \frac{3}{w} \partial_w \gamma_{\mu\nu} = -2\kappa \delta(w - \mathcal{L}) \left(S_{\mu\nu} - \frac{1}{3} \eta_{\mu\nu} S - \frac{2}{\kappa} \partial_{\mu\nu} \zeta \right) \quad (5.1)$$

and to solve it by means of retarded propagators. As is now well known this calculation leads to small corrections to Newton's law of gravity, see [1-2].

We shall recover here this result in a slightly different way which shows that it stems from a particular choice of boundary condition far away from the brane.

As we saw in Section I, solving (5.1) in a distributional sense amounts, in the static case at least, to choosing the solution of (3.3) which converges far from the brane, that is to choosing $e_{\mu\nu}^{(2)}(\vec{k}) = 0$ in (3.11). With this choice of boundary condition at $w \rightarrow +\infty$, the junction conditions (3.12) completely determine the remaining polarization tensor $e_{\mu\nu}^{(1)}(\vec{k})$. The bulk metric is then known as :

$$\gamma_{\mu\nu}(\vec{r}, w) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} e^{i\vec{k} \cdot \vec{r}} \hat{\gamma}_{\mu\nu}(\vec{k}, w) \quad , \quad \hat{\gamma}_{\mu\nu}(\vec{k}, w) = \frac{\kappa M}{3\mathcal{L}(2\pi)^{\frac{3}{2}}} w^2 \frac{K_2(kw)}{k\mathcal{L} K_1(k\mathcal{L})} c_{\mu\nu} \quad (5.2)$$

with $c_{00} = 2$, $c_{0i} = 0$ and $c_{ij} = \delta_{ij} - k_i k_j / 4\pi k^2$ and where $K_\nu(z)$ is the modified Bessel function defined as $K_\nu(z) = i^{\frac{\nu}{2}} e^{i\nu \frac{\pi}{2}} H_\nu^{(1)}(iz)$. Near the brane this metric reduces to, setting $u = \frac{w}{\mathcal{L}} - 1$

$$\hat{\gamma}_{\mu\nu}(\vec{k}, w) = \frac{\kappa M \mathcal{L}}{3(2\pi)^{\frac{3}{2}}} \left\{ \frac{K_2(k\mathcal{L})}{k\mathcal{L} K_1(k\mathcal{L})} - u - \frac{k\mathcal{L}}{4} u^2 \left[\frac{K_2(k\mathcal{L})}{K_1(k\mathcal{L})} - 3 \frac{K_0(k\mathcal{L})}{K_1(k\mathcal{L})} \right] + \mathcal{O}(u^3) \right\} c_{\mu\nu} \quad (5.3)$$

(which implies in particular $\partial_w \gamma_{00}|_\Sigma = -\frac{2\kappa M}{3} \delta_3(\vec{r})$ in accordance with (3.10). The appearance of Dirac distributions in the expansion of $\gamma_{\mu\nu}(\vec{r}, w)$ does not however necessarily mean that $\gamma_{\mu\nu}(\vec{r}, w)$ is singular at $\vec{r} = 0$ as the sum may be regular, as in the scalar field example treated in Section 2).

Let us now concentrate on the h_{00} component of the metric on the brane (which is the same in the x^μ coordinates and the harmonic coordinates \tilde{x}^μ). With ζ given by (3.9), and hence $\hat{\zeta} = -\frac{\kappa M}{6k^2} (2\pi)^{-\frac{3}{2}}$, it reads

$$\hat{h}_{00}(\vec{k}) = \hat{\hat{h}}_{00}(\vec{k}) = \hat{\gamma}_{00}|_\Sigma + 2\frac{\hat{\zeta}}{\mathcal{L}} = \frac{\kappa M}{k^2 \mathcal{L} (2\pi)^{\frac{3}{2}}} \left[1 - \frac{2k\mathcal{L}}{3} \frac{K_0(k\mathcal{L})}{K_1(k\mathcal{L})} \right]. \quad (5.4)$$

Taking the Fourier transform and integrating over angles we obtain, setting $\alpha = r/\mathcal{L}$

$$h_{00}(\vec{r}) = \frac{\kappa M}{4\pi \mathcal{L}} \frac{1}{r} \left(1 + \frac{4\pi}{3} \mathcal{K}_\alpha \right) \quad \text{with} \quad \mathcal{K}_\alpha = \lim_{\epsilon \rightarrow 0} \int_0^{+\infty} du \sin(u\alpha) \frac{K_0(u)}{K_1(u)} e^{-\epsilon u}. \quad (5.5)$$

It is a (fairly) straightforward exercise to see that $\lim_{\alpha \rightarrow 0} \mathcal{K}_\alpha = \alpha^{-1} = \mathcal{L}/r$ and that $\lim_{\alpha \rightarrow \infty} \mathcal{K}_\alpha = \pi/2\alpha^2 = \pi(\mathcal{L}/r)^2/2$. We hence recover that at short distances the correction to Newton's law is in \mathcal{L}/r , whereas as distances large compared with the characteristic scale \mathcal{L} of the anti-de Sitter bulk the correction is reduced by another \mathcal{L}/r factor, in agreement with [1-2]

$$\lim_{r/\mathcal{L} \rightarrow \infty} h_{00}(\vec{r}) = \frac{2GM}{r} \left[1 + \frac{2}{3} \left(\frac{\mathcal{L}}{r} \right)^2 \right]. \quad (5.6)$$

VI. Conclusions

We showed in this paper that the commonly accepted $1/r^2$ correction to Newton's law of the Randall-Sundrum scenario arises from the fact that one imposes the bulk perturbations to be bounded in a conformally Minkowskian coordinate system. In a Gaussian normal coordinate system, which differs more and more from a conformally Minkowskian one as one goes further and further away from the brane, one can have bounded perturbations together with a strictly Newtonian potential on the brane (this is the linearized Chamblin-Hawking-Reall solution).

A question therefore arises : how can one disentangle gauge effects in order to allow only for *geometrically* bounded perturbations in the anti-de Sitter bulk ? We think that this question can be answered by a careful analysis of the asymptotics of anti-de Sitter spacetime, using Schwarzschild-like coordinates adapted to its universal covering.

Another question one may ask is : when considering an extended (static source) to the Chamblin-Hawking-Reall metric, what is the solution like inside the source ? Do we obtain a (unstable) curvature singularity when its size tends to zero as in the Chamblin-Hawking-Reall solution which allows us to reject this solution on these grounds ? This is not completely clear as, first, a linear analysis is no longer sufficient and, second, it is known that when the energy density becomes high enough in the brane Einstein equations cannot hold (see e.g. [8]). We leave this problem to another work [9].

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Appendix

We show here the equivalence between the metric (3.1) (3.2) (4.5) and the metric (4.6) outside the source.

Consider the perturbed anti-de Sitter metric in conformally Minkowskian coordinates $X^A = \{x^\mu = [x^0, \vec{r} = (x^i)], w\}$

$$ds^2|_5 = \left(\frac{\mathcal{L}}{w}\right)^2 (\eta_{AB} + \gamma_{AB}) \quad (A.1)$$

with $\gamma_{Aw} = \gamma_\mu^\mu = \partial_\rho \gamma_\mu^\rho = 0$, with $\gamma_{\mu\nu}(x^\mu, w) = \int \frac{d^3 k}{(2\pi)^{\frac{3}{2}}} e^{i\vec{k} \cdot \vec{r}} \hat{\gamma}_{\mu\nu}(\vec{k}, w)$ and

$$\hat{\gamma}_{\mu\nu}(\vec{k}, w) = i \frac{\kappa\pi M}{12\mathcal{L}} \frac{1}{(2\pi)^{\frac{3}{2}}} w^2 \left[H_0^{(1)}(ik\mathcal{L}) H_2^{(2)}(ikw) - H_0^{(2)}(ik\mathcal{L}) H_2^{(1)}(ikw) \right] c_{\mu\nu} \quad (A.2)$$

where $H_\nu^{(a)}(z)$ are Hankel functions of order ν , of kind (a) and argument z , and where $c_{00} = 2$, $c_{0i} = 0$ and $c_{ij} = \delta_{ij} - k_i k_j / 4\pi k^2$. As an easy manipulation of Bessel functions

shows, this metric reduces near $w = \mathcal{L}$ to

$$\begin{aligned}\gamma_{00}(x^\mu, w) &= \frac{\kappa M}{3\pi\mathcal{L}} \frac{1}{r} + \frac{\kappa M\mathcal{L}}{3} \delta_3(\vec{r}) \left[1 - \left(\frac{w}{\mathcal{L}} \right)^2 \right] + \dots \\ \gamma_{0i}(x^\mu, w) &= 0 \\ \gamma_{ij}(x^\mu, w) &= \frac{\kappa M}{12\pi\mathcal{L}} \frac{1}{r} \left(\delta_{ij} + \frac{x_i x_j}{r^2} \right) + \frac{\kappa M\mathcal{L}}{6} \left(\delta_{ij} \delta_3(\vec{r}) + \frac{1}{4\pi} \partial_{ij} \frac{1}{r} \right) \left[1 - \left(\frac{w}{\mathcal{L}} \right)^2 \right] + \dots\end{aligned}\tag{A.3}$$

The brane is defined by $w = \mathcal{L} + \zeta$ with $\zeta = -\kappa M/24\pi r$ and the metric on the brane is $ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu$ with $h_{\mu\nu} = \gamma_{\mu\nu}|_{w=\mathcal{L}} - 2\zeta\eta_{\mu\nu}/\mathcal{L}$. As shown in the main text this metric on the brane is nothing but the linearized Schwarzschild solution which can be written in the familiar form $\tilde{h}_{00} = \kappa M/4\pi\mathcal{L}r$, $\tilde{h}_{0i} = 0$, $\tilde{h}_{ij} = \delta_{ij}\kappa M/4\pi\mathcal{L}r$ when going to harmonic coordinates $x^\mu \rightarrow \tilde{x}^\mu$ on the brane (N.B. : in both coordinate systems the (00) components are the same, $h_{00} = \tilde{h}_{00}$).

Let us now perform an infinitesimal change of coordinates $X^A \rightarrow \bar{X}^A = X^A + \bar{\epsilon}^A$. If we choose

$$\bar{\epsilon}^4 = -\frac{w\zeta}{\mathcal{L}} \quad , \quad \bar{\epsilon}_\mu = -\frac{\mathcal{L}}{2} \partial_\mu \zeta \left[1 - \left(\frac{w}{\mathcal{L}} \right)^2 \right]\tag{A.4}$$

then the new coordinates will be Gaussian normal, that is the brane will be located at $\bar{w} = \mathcal{L}$ and the metric will read $ds^2|_5 = \left(\frac{\mathcal{L}}{\bar{w}} \right)^2 (\eta_{AB} + \bar{\gamma}_{AB}) d\bar{X}^A d\bar{X}^B$ with

$$\begin{aligned}\bar{\gamma}_{00} &= \frac{\kappa M}{4\pi\mathcal{L}} \frac{1}{r} + \frac{\kappa M\mathcal{L}}{3} \delta_3(\vec{r}) \left[1 - \left(\frac{w}{\mathcal{L}} \right)^2 \right] + \dots \\ \bar{\gamma}_{0i} &= 0 \\ \bar{\gamma}_{ij} &= \frac{\kappa M}{12\pi\mathcal{L}} \frac{1}{r} \left(2\delta_{ij} + \frac{x_i x_j}{r^2} \right) + \frac{\kappa M\mathcal{L}}{6} \delta_3(\vec{r}) \delta_{ij} \left[1 - \left(\frac{w}{\mathcal{L}} \right)^2 \right] + \dots\end{aligned}\tag{A.5}$$

We now set $\frac{\bar{w}}{\mathcal{L}} = e^{\frac{y}{\mathcal{L}}}$, and we also go to harmonic coordinates \tilde{x}^μ on the brane, that is we perform the infinitesimal change of coordinates $\bar{x}^\mu = \tilde{x}^\mu - \tilde{\epsilon}^\mu$ with $\tilde{\epsilon}_0 = 0$ and $\tilde{\epsilon}_i = \frac{\kappa M}{24\pi\mathcal{L}} \frac{x_i}{r}$ so that we have $\partial_\mu (\tilde{h}_\nu^\mu - \delta_\nu^\mu \tilde{h}) = 0$. The bulk metric then reads

$$ds^2|_5 = dy^2 + e^{-2\frac{y}{\mathcal{L}}} (\eta_{\mu\nu} + \tilde{\gamma}_{\mu\nu}) d\tilde{x}^\mu d\tilde{x}^\nu\tag{A.6}$$

with

$$\begin{aligned}\tilde{\gamma}_{00} &= \frac{\kappa M}{4\pi\mathcal{L}} \frac{1}{r} - \frac{2\kappa M}{3} \delta_3(\vec{r}) y \left(1 + \frac{y}{\mathcal{L}} \right) + \mathcal{O}(y^3) \\ \tilde{\gamma}_{0i} &= 0 \\ \tilde{\gamma}_{ij} &= \frac{\kappa M}{4\pi\mathcal{L}} \frac{1}{r} \delta_{ij} - \frac{\kappa M}{3} \delta_3(\vec{r}) y \left(1 + \frac{y}{\mathcal{L}} \right) \delta_{ij} + \mathcal{O}(y^3)\end{aligned}\tag{A.7}$$

Setting $\kappa/\mathcal{L} = 8\pi G$ we therefore see that, everywhere outside the source, that is for $\vec{r} \neq 0$ the metric (A.7) is nothing but the linearized Chamblin-Hawking-Reall solution.

We showed the equivalence of the two metrics near the brane. But an iteration of the bulk equations (3.3) would allow us to extend the transformation to the whole bulk (see

[8] for an example of such a procedure). A important point to note is that, see (A.4), the transformation cannot be infinitesimal for large w . This is not surprising as conformally Minkowskian and Gaussian normal coordinates differ more and more as we go away from the brane. This explains also why the two metrics look so different and why one is even divergent when the other is perfectly well behaved, at least outside the source.

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